Tutorial 8

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1 Question 1: §7.3 Q10

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $v : [c, d] \to \mathbb{R}$ be differentiable on $[a, b]$ with $v([c, d]) \subset [a, b]$. If we define $G(x) = \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c, d]$.

Proof. First, by Theorem 7.3.14, $G(x)$ is well-defined on $[c, d]$. Let $F(y) = \int_a^y f(y)\,\mathrm{d}y$. Then by Fundamental Theorem of Calculus, $F'(y) = f(y)$. Since $G(x) = F(v(x)) = F \circ v(x)$ and F, v are differentiable, $G'(x) = F'(v(x))v'(x)$ by Theorem 6.1.6 Chain Rule, which is

$$G'(x) = f(v(x))v'(x).$$

□

2 Question 2: §7.3 Q13

If $f : \mathbb{R} \to \mathbb{R}$ be continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \int_{x-c}^{x+c} f(t)dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

Proof. For any fixed point $a \in \mathbb{R}$, $g(x) = \int_{x-c}^{x+c} f(t)dt = \int_a^{x+c} f(t)dt - \int_a^{x-c} f(t)dt = g_+(x) - g_-(x)$ by Fundamental Theorem of Calculus. Since $g_+(x) = \int_a^{x+c} f(t)\,\mathrm{d}t$, from Question 1, $g'_+(x) = f(x + c)$. By same reason, $g'_-(x) = f(x - c)$. Thus, $g'(x) = g'_+(x) - g'_-(x) = f(x + c) - f(x - c)$. □

3 Question 3: §7.3 Q22

Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function and let sgn be the signum function. Show that the composite function sgn $\circ\, h$ is not Riemann integrable on $[0, 1]$.

Proof. From above,

$$g(x) = \operatorname{sgn} \circ h = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is a real function on $[0, 1]$, actually called Dirichlet function.

Suppose f is Riemann integrable with its integral L. It is clear that $0 \le L \le 1$. If $L \neq 1$, let $\epsilon = \frac{1-L}{2}$. For any $\delta > 0$, let $P = \{[x_{i-1}, x_i]; t_i \in \mathbb{Q} \cap [x_{i-1}, x_i]\}$ with its norm $\|P\| \le \delta$. Then $|S(f;P) - L| = |1 - L| > \epsilon = \frac{1-L}{2}$, which is a contradiction.

If $L = 1$, let $\epsilon = \frac{L}{2}$. For any $\delta > 0$, let $P = \{[x_{i-1}, x_i]; t_i \in \mathbb{Q}^c \cap [x_{i-1}, x_i]\}$ with its norm $\|P\| \le \delta$. Then $|S(f;P) - L| = |L| > \epsilon = \frac{L}{2}$, which is a contradiction.

Hence it is not Riemann integrable. □

4 Question 4: §7.4 Q7

(a) Prove that if $g(x) = 0$ for $0 \le x \le \frac{1}{2}$ and $g(x) = 1$ for $\frac{1}{2} < x \le 1$, then the Darboux integral of g on $[0, 1]$ is equal to $\frac{1}{2}$.

(b) Does the conclusion hold if we change the value of g at the point $\frac{1}{2}$ to 13?

Proof. (a) Let $P_n = \{[x_{i-1}, x_i]\}_{i=1}^{n} \cup \{[x_{i-1}, x_i]\}_{i=n+1}^{2n}$, where $x_n = \frac{1}{2}$ and $|x_i - x_{i-1}| = \frac{1}{2n}$ for all i.

Hence

$$L(f;P_n) = \sum_{i=1}^{2n} m_i(x_i - x_{i-1}) = 0 \cdot \sum_{i=1}^{n+1}(x_i - x_{i-1}) + 1 \cdot \sum_{i=n+2}^{2n}(x_i - x_{i-1}) = \frac{1}{2} - \frac{1}{2n}$$

and

$$U(f;P_n) = \sum_{i=1}^{2n} M_i(x_i - x_{i-1}) = 0 \cdot \sum_{i=1}^{n}(x_i - x_{i-1}) + 1 \cdot \sum_{i=n+1}^{2n}(x_i - x_{i-1}) = \frac{1}{2}$$

Thus $\int_0^1 f = \frac{1}{2}$.

(b) Yes. Let $P_n = \{[x_{i-1}, x_i]\}_{i=1}^{n} \cup \{[x_{i-1}, x_i]\}_{i=n+1}^{2n}$, where $x_n = \frac{1}{2}$ and $|x_i - x_{i-1}| = \frac{1}{2n}$ for all i.

Hence

$$L(f;P_n) = \sum_{i=1}^{2n} m_i(x_i - x_{i-1}) = 0 \cdot \sum_{i=1}^{n+1}(x_i - x_{i-1}) + 1 \cdot \sum_{i=n+2}^{2n}(x_i - x_{i-1}) = \frac{1}{2} - \frac{1}{2n}.$$

and

$$\begin{aligned} U(f;P_n) &= \sum_{i=1}^{2n} M_i(x_i - x_{i-1}) \\ &= 0 \cdot \sum_{i=1}^{n-1}(x_i - x_{i-1}) + 13(x_n - x_{n-1}) + 13(x_{n+1} - x_n) + 1 \cdot \sum_{i=n+2}^{2n}(x_i - x_{i-1}) \\ &= \frac{1}{2} + \frac{25}{2n} \end{aligned}$$

Hence $\int_0^1 f = \frac{1}{2}$. □

5 Question 5: §7.4 Q12

Let $f(x) = x^2$ for $0 \leq x \leq 1$. For the partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$, calculate $L(f, P_n)$ and $U(f, P_n)$ and show that $L(f) = U(f) = \frac{1}{3}$. Use the formula $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

Proof. By definition,

$$\begin{aligned} L(f, P_n) =& \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ =& \sum_{i=1}^{n} (\frac{i-1}{n})^2 \frac{1}{n} \\ =& \sum_{i=1}^{n} (\frac{i}{n})^2 \frac{1}{n} - \frac{1}{n} \\ =& \frac{1}{6}(2 - \frac{3}{n} + \frac{1}{n^2}) \end{aligned}$$

and

$$\begin{aligned} U(f, P_n) =& \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ =& \sum_{i=1}^{n} (\frac{i}{n})^2 \frac{1}{n} \\ =& \frac{1}{6}(2 + \frac{3}{n} + \frac{1}{n^2}) \end{aligned}$$

Using the fact $\lim_{n\to\infty} L(f; P_n) \leq L(f) \leq U(f) \leq \lim_{n\to\infty} U(f; P_n)$,

$$\frac{1}{3} \leq L(f) \leq U(f) \leq \frac{1}{3}.$$

□